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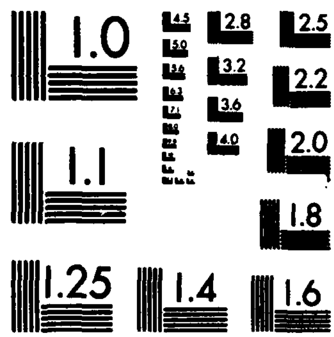

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THE ASYMMETRIC ASSIGNMENT PROBLEM
AND SOME NEW FACETS
OF THE TRAVELING SALESMAN POLYTOPE

by

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Abstract

An assignment (spanning union of node-disjoint dicycles) in a directed graph is called asymmetric if it contains at most one arc of each pair (i,j), (j,i). ~~We~~ describes a class of facets for the asymmetric assignment polytope, associated with certain odd-length closed alternating trails. The inequalities defining these facets are also facet defining for the traveling salesman polytope on the same digraph. Furthermore, this class of facets is distinct from each of the classes identified earlier.

Key words:

- Asymmetric Assignment Problem
- Traveling Salesman Polytope (Facets of)
- Directed Graphs
- Alternating trails

1. Introduction

Let $G = (N,A)$ be the complete digraph on $n = |N|$ nodes with no loops or multiple arcs, and with costs c_{ij} for every arc (i,j). An *assignment* in G is a spanning subgraph that is the node-disjoint union of directed cycles, and the *assignment problem* (AP) is

$$\text{(1)} \qquad \min \sum (c_{ij}x_{ij} : i,j \in N, i \neq j)$$

$$\text{s.t.} \qquad \sum (x_{ij} : j \in N-\{i\}) = 1 \qquad i \in N$$

$$\text{(2)} \qquad \sum (x_{ij} : i \in N-\{j\}) = 1 \qquad j \in N$$

$$\text{(3)} \qquad x_{ij} \in \{0, 1\} \qquad i \in N,\ j \in N - \{i\}.$$

The assignment problem is a frequently used relaxation of the *traveling salesman problem* (TSP), which (on a digraph) asks for a minimum cost directed Hamilton cycle. The TSP can be stated as having the objective function (1) and a constraint set consisting of (2), (3) and

$$\text{(4)} \qquad \sum (x_{ij} : i,j \in S, i \neq j) \leq |S| - 1, \qquad S \subsetneq N,\ 2 \leq |S| \leq \lfloor n/2 \rfloor .$$

An *asymmetric* assignment is one that contains at most one member of every pair of arcs (i,j), (j,i), i.e. contains no directed 2-cycles. The *asymmetric assignment problem* (AAP) has the same objective function (1), and the constraint set (2), (3) and

$$\text{(5)} \qquad x_{ij} + x_{ji} \leq 1 \qquad i, j \in N,\ i \neq j.$$

Clearly, (5) is the subset of (4) corresponding to sets $S \subset N$ such that $|S| = 2$. Thus AAP is also a relaxation of the TSP, stronger (tighter) than AP. In fact, although AAP is closely related to AP, unlike the latter it is *NP*-complete (Garey and Johnson [1980], Sahni [1974]). The *asymmetric assignment (AA) polytope* P is the convex hull of incidence vectors of asymmetric assignments, i.e.

$$P := \text{conv } \{x \in \{0, 1\}^{|A|} \mid x \text{ satisfies } (2), (5)\}.$$

An arc set $S \subset A$ that is the subset of an (asymmetric) assignment will be called an (asymmetric) *partial assignment*. If (2') denotes the system of inequalities obtained from (2) by replacing "=" with "$\leq$", then the incidence vectors of asymmetric partial assignments (APA's for short) are those 0-1 vectors satisfying (2'), (5). The *APA polytope* is

$$\tilde{P} := \text{conv}\,\{x \in \{0,1\}^{|A|} \mid x \text{ satisfies } (2'), (5)\},$$

also called the *monotonization of P*.

The *traveling salesman polytope* P^* is the convex hull of incidence vectors of tours (directed Hamilton cycles), i.e.

$$P^* := \{x \in \{0,1\}^{|A|} \mid x \text{ satisfies } (2), (4)\}.$$

Finally, the *monotone traveling salesman (MTS) polytope* $\tilde{P}^*$ is the convex hull of incidence vectors of partial tours (arc sets that are subsets of a tour, i.e.

$$\tilde{P}^* := \{x \in \{0,1\}^{|A|} \mid x \text{ satisfies } (2'), (4)\}.$$

The polytope $\tilde{P}$, like $\tilde{P}^*$, is easily seen to be full dimensional, i.e. dim $\tilde{P}$ = dim $\tilde{P}^*$ = $n(n-1)$. As to P, since it is contained in the assignment polytope and contains in turn the traveling salesman polytope, and the dimension of these two is known to be the same (Grötschel and Padberg [1985]), namely $n(n-1) - 2n + 1$, it follows that dim P = dim $P^* = n(n-1) - 2n + 1$.

In this paper we describe some new classes of facet inducing inequalities for the traveling salesman polytope P^* defined on a directed graph G. These inequalities define facets of the asymmetric assignment polytope P. They are associated with certain subgraphs of G called closed alternating trails, that correspond to odd holes of the intersection graph of the coefficient matrix of the AAP. Section 2 introduces closed alternating trails and establishes their structural properties. Section 3 uses these properties to identify some classes of facet inducing inequalities for $\tilde{P}$ and $\tilde{P}^*$. In section 4 we prove that for n sufficiently large, these inequalities are also facet inducing for P and P^*. Finally, Section 5 discusses connections with earlier work.

2. Closed Alternating Trails and Their Chords

Let $G^* = (V,E)$ be the intersection graph of the coefficient matrix of the system (2), (5) (or the system (2'), (5)). Then G^* has a vertex for every arc of G; and two vertices of G^* corresponding, say, to arcs (p,q) and (r,s) of G, are joined by an edge of G if and only if either $p = r$, or $q = s$, or $p = s$ and $q = r$. Two arcs of G will be called *G^*-adjacent* if the corresponding vertices of G^* are adjacent. Clearly, there is a 1-1 correspondence between APA's in G and vertex packings (independent vertex sets) in G^*; and therefore the APA polytope $\tilde{P}$ defined on G is identical to the vertex packing polytope defined on G^*.

We define an *alternating trail* in G as a sequence of distinct arcs

$$T = (a_1, \ldots, a_t)$$

such that for $k = 1, \ldots, t-1$, a_k and a_{k+1} are G^*-adjacent, but a_k, a_ℓ, $\ell \neq k+1$, are not; with the possible exception of a_t and a_1. If a_t and a_1 are G^*-adjacent the alternating trail T is *closed*. An arc $a_k = (p,q)$ of T is called *forward* if T meets p before q; *backward* if T meets q before p. The definition of an alternating trail T implies that the direction of the arcs of T alternates between forward and backward, except for pairs a_k, a_{k+1} that form a directed 2-cycle entered and exited by T through the same node; in which case a_k and a_{k+1} are both *forward* or both backward arcs. Notice that T meets a node at most twice, and the number of arcs of T incident from (incident to) any node is at most 2. Two alternating trails,

$$T_1 = ((1,2),(3,2),(3,4),(4,3),(5,3),(5,6),(6,5),(6,7))$$

and

$$T_2 = ((2,1),(2,4),(3,4),(3,2),(5,2)),$$

are shown in Figure 1.

Let G[T] denote the subdigraph of G generated by T; i.e., G[T] has T as its arc set, and the endpoints of the arcs of T as its node set. Further, for any $v \in N$, let $\deg_T^+(v)$ and $\deg_T^-(v)$ denote the outdegree and indegree, respectively, in G[T], of the node v.

The *length* of an alternating trail is the number of its arcs. An alternating trail will be called *even* if it is of even length, *odd* if it is of odd length.

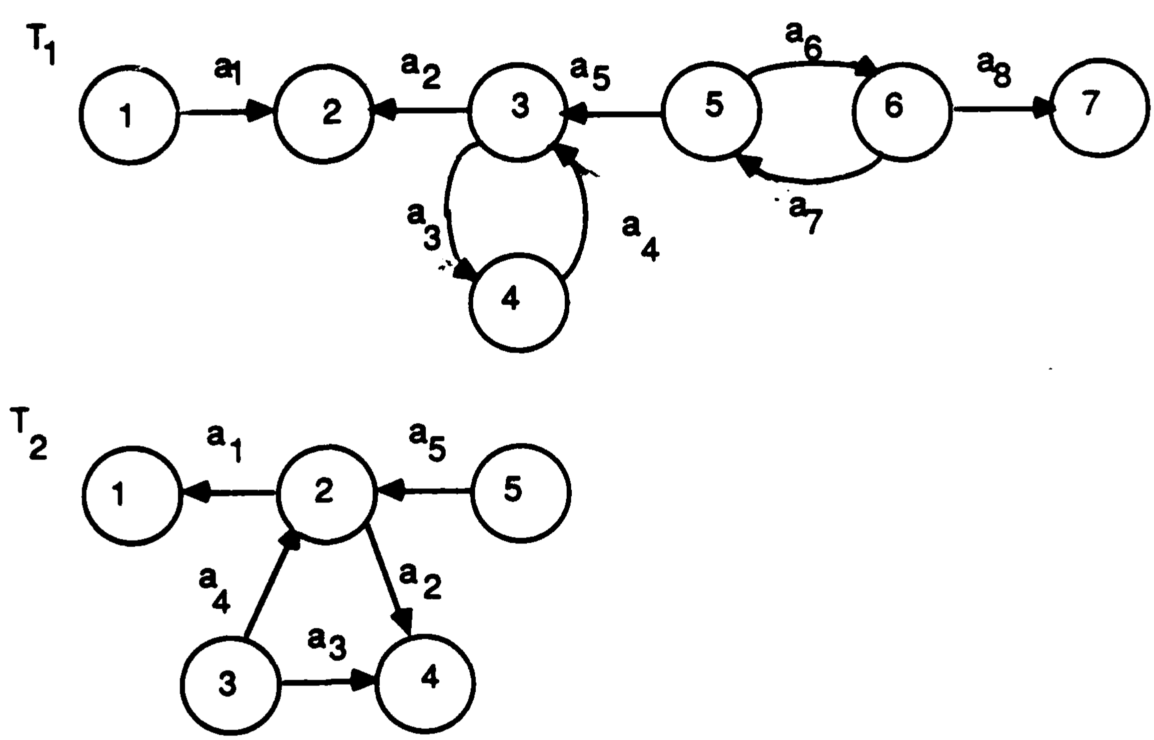


Figure 1

We will be interested in *closed alternating trails* (CAT's for short) of odd length. The reason for this is the following.

Proposition 2.1. There is a 1-1 correspondence between odd CAT's in G and odd holes (chordless cycles) in G^*.

Proof. Follows from the definitions.∎

It is well known (see Padberg [1973]) that the odd holes of an undirected graph give rise to facets of the vertex packing polytope defined on the subgraph generated by the odd hole, and that these facets in turn can be lifted into facets of the polytope defined on the entire graph (see section 4 for details). In order to make the lifting procedure conveniently applicable to the particular vertex packing polytope associated with G^*, we need the structural information concerning adjacency relations on G^* that will be developed in this section.

Let T be a CAT in G. A node of G[T] will be called a *source*, if it is the common tail of two arcs of T; and a *sink,* if it is the common head of two arcs of T. A node of G[T] can thus be a source, or a sink, or both, or none. A node of G[T] that is neither a source nor a sink will be called *neutral.* Several odd CAT's are illustrated in Figure 2. The sources and sinks of $G[T_1]$ are nodes 1, 2 and 2, 4, respectively, while 3 is neutral. $G[T_2]$ has three neutral nodes, 1, 4 and 6, while nodes 2, 3 and 5 are both sources and sinks. $G[T_3]$ has sources 1 and 4, sinks 2, 3 and 4, while 5 is a neutral node.

Proposition 2.2. Let T be an odd CAT of length t, with q neutral nodes. then

(6) $\quad 1 \leq q \leq t/3 \quad$ and q is odd.

Proof. Let $T = (a_1,\ldots,a_t)$. If $q = 0$, the arcs of T alternate between forward and backward, and either a_1 is forward and a_t is backward, or vice versa. But this implies that T is even, a contradiction. Hence $q \geq 1$.

Now suppose $q > t/3$. Then T has more than $2t/3$ arcs incident with neutral nodes and less than $t/3$ arcs not incident with such nodes. Since T

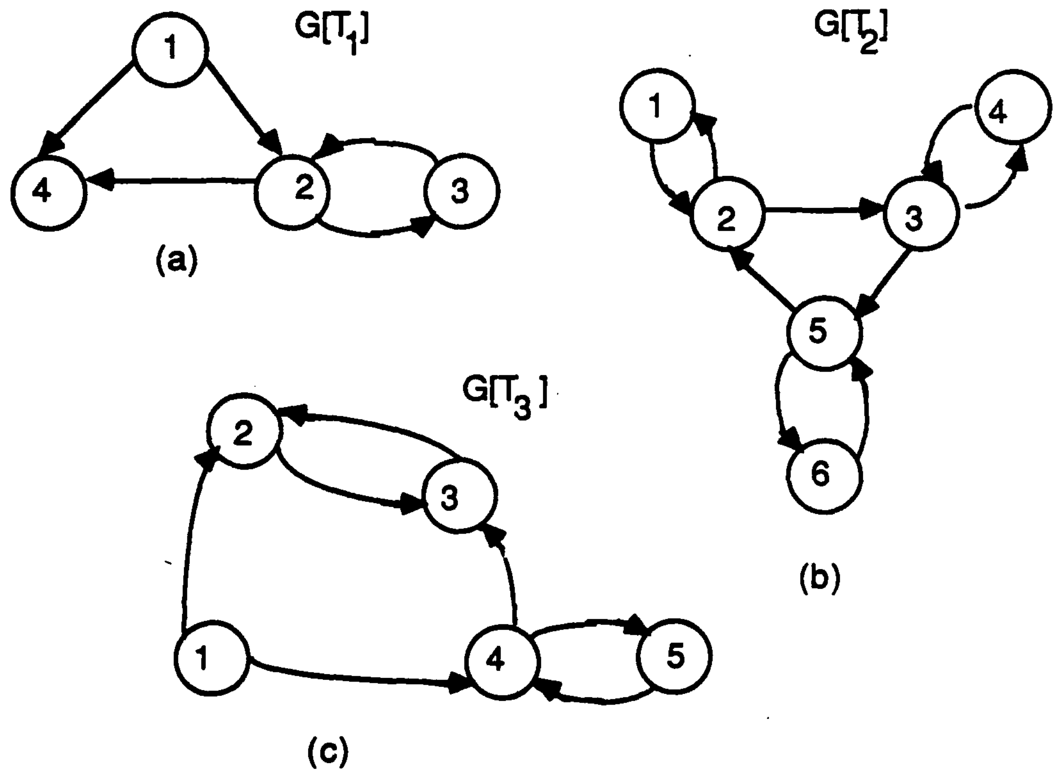


Figure 2

has more than t/3 2-cycles and less than t/3 arcs left to separate them, there exists a pair of 2-cycles with a common node. But such a node has indegree and outdegree greater than 2 in T, contrary to the definition of an alternating trail. Thus $q \leq t/3$.

We have seen above that for $q = 0$ T is even. For arbitrary $q > 0$, the alternating sequence of forward and backward arcs is interrupted q times by a repetition of type (forward or backward). Thus for q even, the number of repetitions cancel out, and the fact that a_1 and a_t are of opposite directions makes for an even T. For q odd, all but one of the repetitions cancel out, and T must be odd.∎

Proposition 2.3. Let T be an odd CAT of length t, with s sources, u sinks and w 2-cycles. Then

(7) $\qquad s + u + w = t.$

Proof. T has 2w arcs belonging to 2-cycles and t − 2w arcs not belonging to 2-cycles. Every arc of a 2-cycle has either a source for its tail or a sink for its head, but not both. Every arc not belonging to a 2-cycle has both a source for its tail and a sink for its head. Hence

$$\begin{aligned} s + u &= 1 \times w + 2 \times (t-2w)/2 \\ &= t - w. \end{aligned}$$ ∎

In the sequel we will denote by $\mathcal{F}$ the family of APA's in G.

Proposition 2.4. Let T be an odd CAT of length t. Then

(8) $$\max_{S \in \mathcal{F}} |S \cap T| = (t-1)/2.$$

Furthermore, for any pair of G^*-adjacent arcs a_k, a_{k+1} of T (with t + 1 = 1), there exists $S \in \mathcal{F}$, with $a_k \notin S$, $a_{k+1} \notin S$, and $|S \cap T| = (t-1)/2$.

Proof. For any $S \in \mathcal{F}$, $S \cap T$ contains no pair of G^*-adjacent arcs. The largest such set clearly has cardinality $\lfloor t/2 \rfloor = (t-1)/2$. Further, for any such $S \in \mathcal{F}$, $|T \setminus S| = (t+1)/2$, hence $T \setminus S$ contains a pair of G^*-adjacent arcs of T; and for any pair a_k, a_{k+1} of G^*-adjacent arcs of T, there exists $S \in \mathcal{F}$ containing $(t-1)/2$ arcs of $T \setminus \{a_k, a_{k+1}\}$.∎

A *chord* of a CAT T is an arc $a \in A \setminus T$ joining two nodes of G[T]. If T is odd and a = (u,v), a divides T into two disjoint subtrails, one odd (T_1) and one even (T_2), each of which connects u to v. We distinguish between three types of chords. A chord (u,v) is of

- *type 1* if it joins a source to a sink (i.e. $\deg_T^+(u) = \deg_T^-(v) = 2$);

- *type 2* if it joins a source to a neutral node, or a neutral node to a sink (i.e. $\deg^+_T(u) + \deg^-_T(v) = 3$) and the *even* subtrail T_2 connecting u to v has its first arc incident *from* u and its last arc incident *to* v;
- *type 3* in all other cases.

Figure 3 shows the odd CAT

$$T_1 = ((1,2),(3,2),(3,4),(4,3),(5,3)(5,6),(1,6))$$

with its chords of type 1 (1,3), (3,6), (5,2) in shaded lines.

As T_1 has no chords of type 2, all other chords (not shown) are of type 3.

Figure 4 shows the odd CAT

$$T_2 = ((1,2),(3,2),(2,3),(2,4),(5,4),(4,5),(4,6),(7,6),(6,7),(6,8),(1,8))$$

with its chords of type 1 in shaded lines, ((1,4), (1,6), (2,6), (2,8), (4,2), (4,8), (6,2), (6,4)), and its chords of type 2 in checkered lines , ((1,5), (2,7), (3,6), (4,3), (5,2), (5,8), (6,5) and (7,4)). All other chords (not shown) are of type 3.

Finally, of the three odd CAT's of figure 2, T_1 has only chords of type 3; whereas T_2 has three chords of type 1, (2,5), (3,2) and (5,3), and T_3 has two chords of type 1, (1,3) and (4,2); all remaining chords are of type 3.

Proposition 2.5. Let T be an odd CAT of length t and for k = 1, 2, 3, let C_k be the set of chords of T of type k. Then

$$(9) \qquad \max_{S \in \mathcal{T}:\, c \in S} |S \cap (T \cup \{c\})| = \begin{cases} (t-1)/2 & c \in C_1 \cup C_2 \\ (t+1)/2 & c \in C_3 \end{cases}$$

Proof. Let c = (u,v), and let $T = T_1 \cup T_2$, where T_1 and T_2 are the two subtrails of T connecting u to v. Since T is odd, we may assume w.l.o.g. that T_1 is odd and T_2 is even.

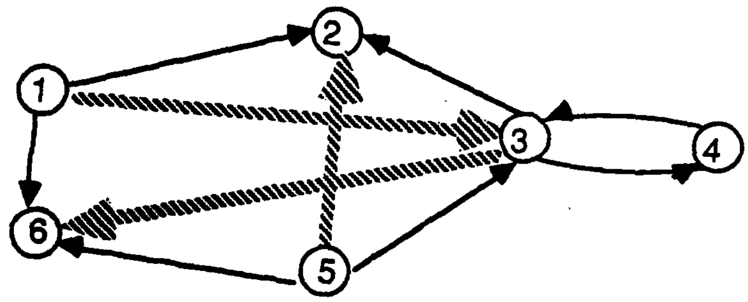


Figure 3

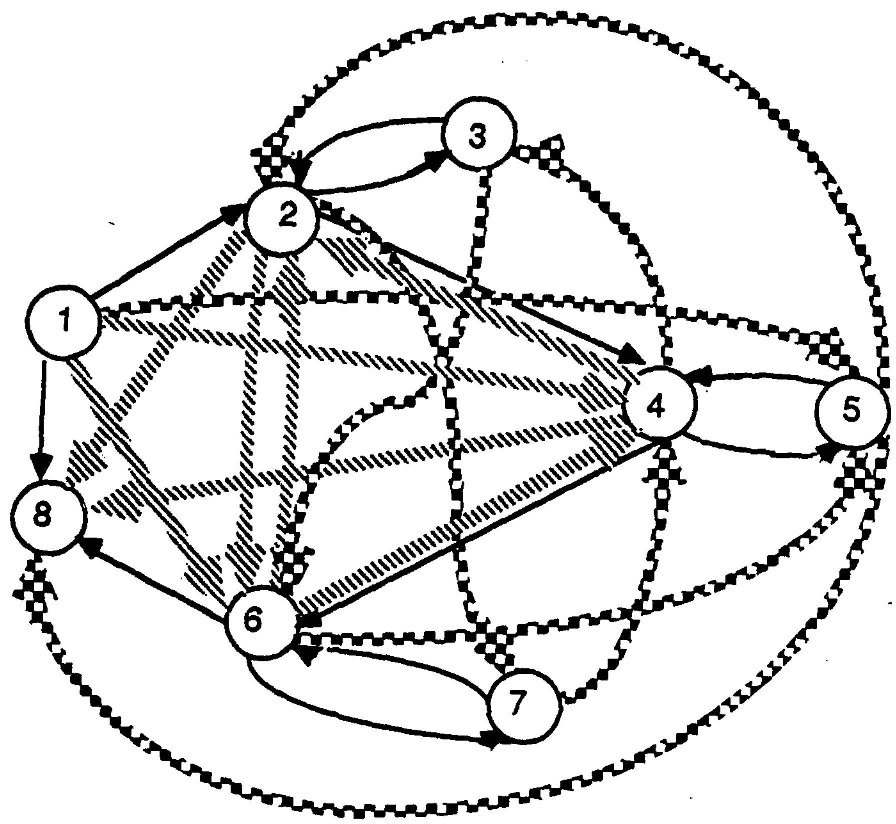


Figure 4

If $c \in C_1$, then $c \in S$ implies that S cannot contain the first and last arcs of T_1 and T_2. Hence the maximum number of arcs of T_1 and T_2 contained by any such S is $(|T_1|-1)/2$ and $(|T_2|-2)/2 = |T_2|/2 - 1$, respectively, and the maximum of the expression on the left-hand side of (9) is $(|T_1|-1)/2$ (arcs of T_1) + $|T_2|/2 - 1$ (arcs of T_2) + 1 (the arc c) = $(|T_1|+|T_2|-1)/2 = (t-1)/2$.

If $c \in C_2$, then $c \in S$ implies that S cannot contain the first and the last arc of T_2 (by definition of C_2, the last arc of T_2 is incident *to* v) and S cannot contain *both* the first and last arcs of T_1. Then the maximum number of arcs of T_2 contained by any $S \in \mathcal{F}$ that contains c is, as in the earlier case, $|T_2|/2 - 1$, and the maximum number of arcs of T_1 is $(|T_1|-1)/2$, also like in the earlier case. Thus the maximum of the left-hand side of (9) is again $(t-1)/2$.

Finally, if $c \in C_3$, S can contain, besides c, $(|T_1|-1)/2$ arcs of T_1 and $|T_2|/2$ arcs of T_2, and the maximum of the left-hand side of (9) is $(|T_1|-1)/2 + |T_2|/2 + 1 = (t+1)/2$. ‖

Proposition 2.6. Let T be an odd CAT of length t and let C_1 be the set of chords of T of type 1. Then

$$(10) \qquad \max_{S \in \mathcal{F}} |S \cap (T \cup C_1)| = (t-1)/2$$

Proof. Let $S^* \in \mathcal{F}$ be such that $|S^* \cap (T \cup C_1)| = \max |S \cap (T \cup C_1)|$. W.l.o.g. we may assume that for every neutral node v of G[T], $S^* \cap T$ contains an arc incident with v. Indeed, should this not be the case for some v, one can always replace the arc of $S^* \cap T$ incident with the (unique) node adjacent in T to v, with one of the two arcs incident with v in order to obtain an APA S^0 such that $|S^0 \cap (T \cup C_1)| = |S^* \cup (T \cup C_1)|$.

Let s, u and q denote the number of sources, sinks and neutral nodes, respectively, of G[T], and let w stand for the number of 2-cycles of T. Let $S^* \cap (T \cup C_1) = S_1 \cup S_2$, where S_1 is the set of arcs in $S^* \cap T$ incident with a neutral node, and $S_2 = S^* \cap (T \cup C_1) \setminus S_1$. By assumption, $|S_1| = q$. From Proposition 2.3 and the fact that $q \leq w$,

$$s + u \leq t - q.$$

Every arc in S_2 has a source for its tail and a sink for its head; every arc in S_1 has either a source for its tail or a sink for its head, but not both; and no two arcs in $S_1 \cup S_2$ have a common tail or a common head. Hence

$$\begin{aligned} |S_1 \cup S_2| &\leq q + \frac{1}{2}(s+u-q) \\ &\leq q + \frac{1}{2}(t-2q) \\ &= \frac{1}{2}t \end{aligned}$$

or, since t is odd, $|S^* \cap (T \cup C_1)| \leq (t-1)/2$. ▌

Proposition 2.7. Let G be a complete digraph. Let T be an odd CAT of length t, and let C_1 be the set of chords of T of type 1. Then

(11) $$\max_{S \in \mathcal{F}} |S \cap (T \cup C_1 \cup \{a\})| = (t+1)/2, \ \forall\, a \in A \setminus (T \cup C_1)$$

Proof. Since there exists $S \in \mathcal{F}$ such that $|S \cap (T \cup C_1)| = (t-1)/2$, if a is not a chord then clearly there exists $S \in \mathcal{F}$ such that $|S \cap (T \cup C_1 \cup \{a\})| = (t-1)/2 + 1 = (t+1)/2$. If a is a chord of type 3, the existence of such S follows from Proposition 2.5. Also, in both cases $(t+1)/2$ is the maximum size of $S \cap (T \cup C_1 \cup \{a\})$, or else we have a contradiction to Proposition 2.6.

Now let $a \in C_2$, $a = (u,v)$. As before, let T_1 and T_2 be the two subtrails of T connecting u to v, with T_1 odd and T_2 even. Let $S \in \mathcal{F}$ be such that $a \in S$ and $|S \cap (T \cup \{a\})| = (t-1)/2$. Then S contains $(t-3)/2$ arcs of T, namely $(|T_1|-1)/2$ arcs of T_1 and $|T_2|/2 - 1$ arcs of T_2. Thus T_2 has two G^*-adjacent arcs not contained in S, say a_k and a_{k+1}, that have either a common tail or a common head. Suppose node w is the common tail of a_k and a_{k+1}, with $\deg_S^+(w) = 0$. (An analogous reasoning holds in case of a common head.)

W.l.o.g., assume S contains the arc of T_1 incident from v, say a_ℓ (there is always some set S with this property among those $S \in \mathcal{F}$ such that $a \in S$ and $|S \cap T \cup \{a\}| = (t-1)/2$). Since $T_1 \setminus \{a_\ell\}$ is even and has $(|T_1|-3)/2$ arcs in S, of which the first and last arc cannot belong to S (since the first arc is G^*-adjacent to a and the last one is G^*-adjacent to a_ℓ), it has two G^*-adjacent arcs not contained in S, say a_m, a_{m+1}, with a common tail s and $\deg^+_S(s) = 0$, or a common head s, and $\deg^-_S(s) = 0$. W.l.o.g., suppose the latter case holds. Since G is complete, it has an arc $c_1 = (w,s)$, and this arc is a chord of T of type 1. Then $S^\circ := S \cup \{c_1\}$ is an APA that contains $(t+1)/2$ arcs of $T \cup \{a,c_1\}$. ∎

Remark. If G is not complete, (11) may not hold for some $a \in C_2$. It still holds for all $a \in A \setminus (T \cup C_1 \cup C_2)$.

3. Facets of the Monotone Polytopes $\tilde{P}$ and $\tilde{P}^*$

We are now ready to characterize the class of facet inducing inequalities of the APA polytope $\tilde{P}$ associated with odd CAT's. We consider first the subgraph generated by an odd CAT.

As mentioned in Section 2, $\tilde{P}$ is the same as the vertex packing polytope defined on G^*. Further, every odd CAT of G corresponds to an odd hole of G^*. It is well known (Padberg [1973]) that odd holes of an undirected graph give rise to facet inducing inequalities for the vertex packing polyhedron defined on the subgraph generated by the odd hole. Nevertheless, because of its simplicity and its usefulness in subsequent developments, we give a direct proof of this result for our case.

For any $S \subseteq A$, we denote $x(S) = \sum(x_{ij} : (i,j) \in S)$.

Proposition 3.1. Let T be an odd CAT of length t, and let $\tilde{P}$ be the APA polytope defined on G[T]. Then the inequality

(12) $x(T) \leq (t-1)/2$

defines a facet of $\tilde{P}(G[T])$.

Proof. From Proposition 2.4, (12) is satisfied by all $x \in \tilde{P}(G[T])$. Let $T := (a_1, \ldots, a_t)$. Define $x^1 \in \tilde{P}(G[T])$ by $x^1_i = 1$ if i is odd and $1 \leq i < t$, $x^1_i = 0$ otherwise, and for $k = 2, \ldots, t$, define x^k by $x^k_i = x^{k-1}_{i-1}$, $i = 1, \ldots, t$, with $i - 1 = t$ for $i = 1$. Then the vectors x^k, $k = 1, \ldots, t$ form the rows of a circulant matrix of order t with $(t-1)/2$ 1's in every row (and every column), known to be nonsingular. Hence the t points x^k, $k = 1, \ldots, t$, which are clearly contained in $\tilde{P}(G[T]) \cap \{x \mid x(T) = (t-1)/2\}$, are affinely independent. Thus (12) defines a facet of $\tilde{P}(G[T])$.∎

Corollary 3.2. The inequality (12) defines a facet of $\tilde{P}^*(G[T])$.

Proof. Since $\tilde{P}^*(G[T]) \subset \tilde{P}(G[T])$, the inequality (12) is valid for $\tilde{P}^*(G[T])$. Since $\tilde{P}^*(G[T])$ is full dimensional, (12) does not define an improper face. Finally, each of the t affinely independent points $x^k \in \tilde{P}(G[T])$ used in the proof of Proposition 3.1 is a point of $\tilde{P}^*(G[T])$, i.e. an incidence vector of a partial tour. Thus (12) defines a facet of $\tilde{P}^*(G[T])$.∎

Next we will "lift" the inequality (12) to identify inequalities of the form

(13) $x(T) + \sum(\alpha_{ij}x_{ij} : (i,j) \in A\setminus T) \leq (t-1)/2$

that define facets of $\tilde{P}$. It is a well-known result in combinatorial optimization (see Padberg [1973], Nemhauser and Trotter [1974], Balas and Zemel [1984]) that if (12) defines a facet of $\tilde{P}(G[T])$, there exist integers α_{ij}, $(i,j) \in A \setminus T$, such that (13) defines a facet of $\tilde{P}$. Furthermore, for any ordering $(i(1),j(1)), \ldots, (i(p),j(p))$ of the arc set $A \setminus T$, there exists such a (not necessarily distinct) facet defining inequality, whose coefficients α_{ij} can be obtained by solving a sequence of integer programs. To be more specific, if for $k = 1, \ldots, p$, $G_k = (N_k, A_k)$ is the graph consisting of the arcs in $T \cup \{(i(1),j(1)), \ldots, (i(k),j(k))\}$ and their endpoints, the coefficients of (13) are obtained by setting, for $k = 1, \ldots, p$,

$$\alpha_{i(k)j(k)} = (t-1)/2 - z_{i(k)j(k)}, \tag{14}$$

where

$$
\begin{aligned}
z_{i(k)j(k)} = \max \Sigma \{x_{ij} : (i,j) \in A_k \setminus \{(i(k),j(k))\}\} \\
\Sigma\{x_{ij} : j \in \Gamma_k(i)\} \leq 1 \qquad & i \in N_k \setminus \{i(k)\} \\
\Sigma\{x_{ij} : i \in \Gamma_k^{-1}(j)\} \leq 1 \qquad & j \in N_k \setminus \{j(k)\} \\
x_{i(k)j} = 0,\ j \neq j(k);\ x_{ij(k)} = 0,\ i \neq i(k) \\
x_{ij} \in \{0, 1\},\ (i,j) \in A_k,
\end{aligned}
\tag{15}
$$

and where $\Gamma_k(i)$ and $\Gamma_k^{-1}(i)$ are the sets of successors and predecessors, respectively, of node i in G_k.

It follows from the above definition of the lifting coefficients, that in comparing two inequalities of the form (13), say $(13)_1$ with coefficients α^1_{ij}, and $(13)_2$ with coefficients α^2_{ij}, for any given arc (i_*,j_*) we have $\alpha^1_{i_*j_*} \geq \alpha^2_{i_*j_*}$ if the rank of (i_*,j_*) in the sequence associated with $(13)_1$ is lower than in the sequence associated with $(13)_2$. So a given coefficient has the highest value if the corresponding variable is lifted first, and the lowest value if it is lifted last.

Theorem 3.3. Let $G = (N,A)$ be a complete digraph. Let T be an odd CAT of length t, and let C_1 be the set of chords of T of type 1. Then the inequality

$$x(T \cup C_1) \leq (t-1)/2 \tag{16}$$

defines a facet of the APA polytope $\tilde{P}$ and the MTS polytope $\tilde{P}^*$.

Proof. We lift the inequality (12) by taking the arcs of $A \setminus T$ in any order such that all arcs in C_1 precede all arcs in $A \setminus (T \cup C_1)$. Let $c_1 = (i(1),j(1))$ be the arc in C_1 whose variable is lifted first. Then the maximum of the integer program (15) is $z_{i(1)j(1)} = (t-3)/2$, since from Proposition 2.5

$$\max_{S \in \mathcal{F}: c_1 \in S} |S \cap T| = \max_{S \in \mathcal{F}: c_1 \in S} |S \cap (T \cup \{c_1\})| - 1 = (t-3)/2.$$

Thus $\alpha_{i(1)j(1)} = (t-1)/2 - z_{i(1)j(1)} = 1$.

We claim that $\alpha_{i(k)j(k)} = 1$, $k = 1, \ldots, m$, where $\{(i(1),j(1)), \ldots, (i(m),j(m))\} = C_1$. Suppose the claim is true for $k = 1, \ldots, \ell - 1$, and let $k = \ell \geq 2$. From Propositions 2.5 and 2.6,

$$\max\{|S \cap (T \cup \{(i(1),j(1)), \ldots, (i(\ell),j(\ell))\}| : (i(\ell),j(\ell)) \in S\} = (t-1)/2$$

hence

$$\begin{aligned} z_{i(\ell)j(\ell)} &= \max \{|S \cap (T \cup \{i(1),j(1),\ldots,(i(\ell-1),j(\ell-1))\}| : (i(\ell),j(\ell)) \in S\} \\ &= (t-3)/2 \end{aligned}$$

and thus $\alpha_{i(\ell)j(\ell)} = (t-1)/2 - z_{i(\ell)j(\ell)} = 1$, which proves the claim. Thus the lifting coefficients of all variables corresponding to arcs of C_1 are equal to 1.

Consider now the coefficient of the variable associated with some arc $a \in A \setminus (T \cup C_1)$ that is lifted *first* after the variables corresponding to arcs in C_1. Let $a = (i(m+1),j(m+1))$. Since G is complete, from Proposition 2.7 we have

$$\max \{|S \cap (T \cup \{(i(1),j(1)), \ldots, (i(m+1),j(m+1)\}| : (i(m+1),j(m+1)) \in S\} = (t+1)/2$$

and thus

$$\begin{aligned} z_{i(m+1)j(m+1)} &= \max\{|S \cap (T \cup \{(i(1),j(1)), \ldots, (i(m),j(m))\}| : (i(m+1),j(m+1)) \in S\} \\ &= (t+1)/2 - 1 = (t-1)/2. \end{aligned}$$

Therefore $\alpha_{i(m+1)j(m+1)} = (t-1)/2 - z_{i(m+1)j(m+1)} = 0$. Since the variable associated with the arc a has a coefficient of 0 when it is first in the lifting sequence (among the arcs in $A \setminus (T \cup C_1)$), it has a coefficient of 0 also when it is in any subsequent position in the sequence.

This proves that the lifting coefficients of the arcs $a \in A \setminus T$ are, for any lifting sequence that puts all arcs in C_1 before all arcs in $A \setminus (T \cup C_1)$,

$$\alpha_a = \begin{cases} 1 & a \in C_1 \\ 0 & a \in A \setminus (T \cup C_1) \end{cases}$$

which proves that (16) defines a facet of $\tilde{P}$ and of $\tilde{P}^*$. ∎

The arc sets corresponding to the support (i.e. the set of positive coefficients) of each inequality (16) in the digraphs with 4, 5 and 6 vertices are shown (up to isomorphisms) in Figures 5 and 6, with the arcs of T and C_1 shown in solid and shaded lines, respectively. The number of such inequalities is 24 for a graph on four nodes, 360 for a graph on five nodes, and 3,360 for a graph on six nodes.

It is easy to establish the *Chvatal rank* of the inequalities (16). Chvatal's [1973a] procedure for generating all the inequalities valid for a polyhedron defined as the convex hull of integer points satisfying a given set of linear inequalities $Ax \leq b$ consists of recursively applying the following step: take all undominated positive linear combinations of the inequalities of the current system and add the resulting inequalities to the system after rounding down all coefficients to the nearest integer. The initial system $Ax \leq b$ is said to have rank 0, while the inequalities obtained in the first step of the recursion have rank 1.

Remark 3.4. The inequalities (16) have Chvatal rank 1.

Proof. Each inequality (16) associated with an odd CAT T can be obtained by adding the equations (2) associated with each source and each sink of G[T], and the inequalities associated with each two-cycle of G[T]; then dividing by two the resulting inequality and rounding down all coefficients to the nearest integer. ∎

The inequalities (16) can be expressed in an equivalent form as "$\geq$" inequalities with all lefthand-side coefficients equal to 0 or 1. Indeed, let S be the set of arcs whose tail is a source of G[T], let U be the set of arcs whose head is a sink of G[T], and let W be the set of arcs contained in a two-cycle of T.

Theorem 3.4. A vector $x \in P_I(A)$ satisfies (16) if and only if it satisfies

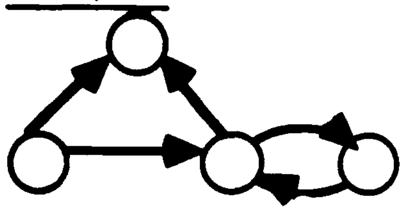
(16') $$x(S \cup U \setminus W) \geq \frac{1}{2}(t+1) - q.$$

Proof. Subtracting from (16) the equations (2) corresponding to every source and every sink of G[T], and multiplying the resulting inequality by - 1 yields (16').∎

There are some inequalities other than the family (16) that can be obtained by lifting the inequality (12), but their description is more cumbersome. The following rules apply to all facet defining inequalities (for $\tilde{P}$ and $\tilde{P}^*$) obtained by sequential lifting from (12). As before, for k = 1, 2, 3, let C_k be the set of chords of type k.

- All variables corresponding to chords in $C_1 \cup C_2$ get a coefficient of 0 or 1, and all remaining variables get a coefficient of 0, irrespective of the lifting sequence.
- If all variables corresponding to chords in C_1 are lifted before all others, then all variables corresponding to chords in C_1 get a coefficient of 1 and all remaining variables get a coefficient of 0 (this is the family (16)).
- If a variable corresponding to a chord $(i,j) \in C_2$ is lifted first, it gets a coefficient of 1; but then the variables corresponding to certain chords in C_1, whose identity depends on (i,j), get a coefficient of 0, as do all the variables corresponding to arcs in $A \setminus (T \cup C_1 \cup \{(i,j)\})$.

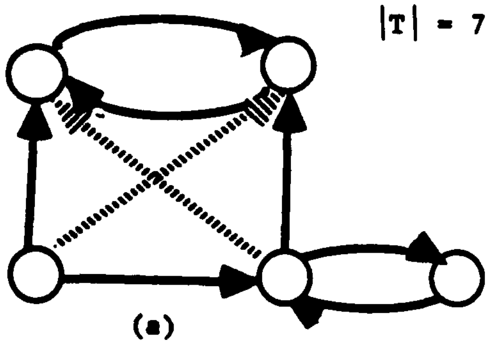
$|N(T)| = 4$



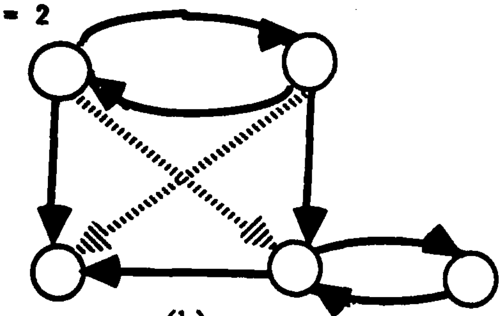
$|T| = 5, \quad |C_1| = 0$

$|N(T)| = 5$

$|T| = 7, \ |C_1| = 2$



(a)

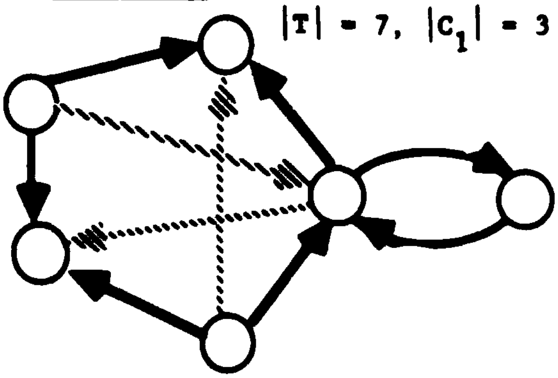


(b)

Figure 5

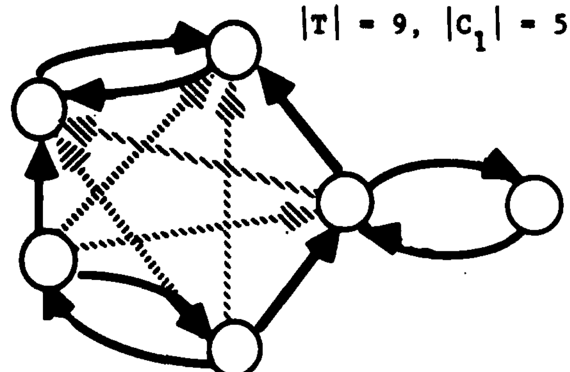
$|N(T)| = 6$

$|T| = 7, \ |C_1| = 3$

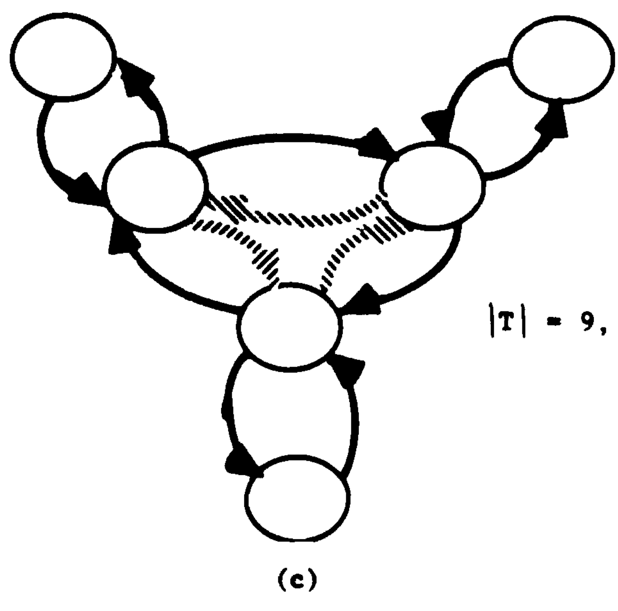


(a)

$|T| = 9, \ |C_1| = 5$



(b)



$|T| = 9, \ |C_1| = 3$

(c)

Figure 6

4. Facets of the Polytopes P and P^*

Theorem 4.1. For $n \geq 6$ and any odd CAT T in the complete digraph G, the inequality (16) defines a facet of the AA polytope P.

Proof. Let

$$F := P \cap \{x \mid x(T \cup C_1) = (t-1)/2\}.$$

We will show that F is a facet of P by proving that dim F = dim P - 1. This in turn will be accomplished by showing that for any linear inequality $\alpha x \leq \alpha_0$ satisfied by all $x \in P$ and satisfied with equality by all $x \in F$, the equation $\alpha x = \alpha_0$ is a linear combination of the equations (2) and $x(T \cup C_1) = (t-1)/2$.

W.l.o.g., assume the nodes of G to be numbered so that $n-1$ is not a source and n is not a sink of G[T]. Such a pair always exists. Define

$$\text{(17)} \qquad \lambda_i = \begin{cases} \alpha_{in} - \alpha_{n-1,n} & i = 1, \ldots, n-1 \\ 0 & i = n \end{cases}$$
$$\mu_j = \begin{cases} \alpha_{n-1,j} & j = 1, \ldots, n-2, n \\ 0 & j = n-1 . \end{cases}$$

We will show that there exists a scalar π_0, which together with the λ_i, μ_j satisfies

$$\text{(18)} \qquad \alpha_{ij} = \begin{cases} \lambda_i + \mu_j & \text{if } (i,j) \notin T \cup C_1 \\ \lambda_i + \mu_j + \pi_0 & \text{if } (i,j) \in T \cup C_1 \end{cases}$$

and

$$\text{(19)} \qquad \alpha_0 = \Sigma(\lambda_i : i \in N) + \Sigma(\mu_j : j \in N) + ((t-1)/2)\pi_0 .$$

For $(i,j) \notin T \cup C_1$, we have to show that

$$\text{(18')} \qquad \alpha_{ij} = \lambda_i + \mu_j = \alpha_{in} + \alpha_{n-1,j} - \alpha_{n-1,n}.$$

This is obviously true for $i = n - 1$, $j \in N - \{n - 1\}$ and for $i \in N - \{n\}$, $j = n$. For $i \neq n - 1$, $j \neq n$, $i \neq j$, consider $x \in F$ such that $x_{ij} = x_{n-1,n} = 1$ and $x_{ni} = x_{j,n-1} = 0$. For $n \geq 6$ such x always exists. Define x' by $x'_{ij} = x'_{n-1,n} = 0$, $x'_{in} = x'_{n-1,j} = 1$, $x'_{k\ell} = x_{k\ell}$ for all other k, ℓ. Then x' is an asymmetric assignment and $x'(T \cup C_1) = (t-1)/2$; i.e. $x' \in F$. By assumption, $\alpha x = \alpha x' = \alpha_0$, hence

$$\alpha x - \alpha x' = \alpha_{ij} + \alpha_{n-1,n} - \alpha_{in} - \alpha_{n-1,j} = 0,$$

i.e. (18') holds.

For $(i,j) \in T \cup C_1$, define

(20) $$\pi_{ij} = \alpha_{ij} - \lambda_i - \mu_j.$$

We will show that the π_{ij} are all equal.

Let $(i,j) \in T \cup C_1$ be such that i is a source and j is a sink of G[T], and there exists a node v of G[T], $i \neq v \neq j$, that is both a source and a sink. Such (i,j) always exists. For every sink ℓ of G[T], $\ell \neq i, j$, choose some $k \neq i, j, \ell$ such that $(k,\ell) \notin T \cup C_1$, $(k,j) \notin T \cup C_1$ (such k obviously exists). Since i is a source and ℓ is a sink of G[T], $(i,\ell) \in T \cup C_1$. Now consider $x \in F$ such that $x_{ij} = x_{k\ell} = 1$ and $x_{jk} = x_{\ell i} = 0$. Define x' by $x'_{ij} = x'_{k\ell} = 0$, $x'_{i\ell} = x'_{kj} = 1$, and $x'_{rs} = x_{rs}$ for all other r, s. Then $x' \in F$, hence $\alpha x = \alpha x' = \alpha_0$ and thus

(21) $$\alpha_{ij} + \alpha_{k\ell} = \alpha_{i\ell} + \alpha_{kj}.$$

Similarly for every source p of G[T], $p \neq i, j$, choose some $q \neq i, j, p$ such that $(p,q) \notin T \cup C_1$, $(i,q) \notin T \cup C_1$ (such q always exists). Since p is a source and j is a sink of G[T], $(p,j) \in T \cup C_1$. Consider now $\bar{x} \in F$ such that $\bar{x}_{ij} = \bar{x}_{pq} = 1$, $\bar{x}_{jp} = \bar{x}_{qi} = 0$. Define $\bar{x}'$ by $\bar{x}'_{ij} = \bar{x}'_{pq} = 0$, $\bar{x}'_{iq} = \bar{x}'_{pj} = 1$, and $\bar{x}'_{rs} = \bar{x}_{rs}$ for all other r, s. Then $\bar{x}' \in F$, $\alpha\bar{x} = \alpha\bar{x}'$, and thus

(22) $$\alpha_{ij} + \alpha_{pq} = \alpha_{iq} + \alpha_{pj}.$$

Substituting into (21) the expression from (20) for α_{ij} and $\alpha_{i\ell}$, and the expression from (18) for α_{kj} and $\alpha_{k\ell}$, we obtain

$$\pi_{ij} + \lambda_i + \mu_j + \lambda_k + \mu_\ell = \pi_{i\ell} + \lambda_i + \mu_\ell + \lambda_k + \mu_j,$$

or $\pi_{ij} = \pi_{i\ell}$.

Similarly, substituting into (22) the expression from (20) for α_{ij} and α_{pj}, and the expression from (18) for α_{iq} and α_{pq}, we get

$$\pi_{ij} + \lambda_i + \mu_j + \lambda_p + \mu_q = \pi_{pq} + \lambda_p + \mu_j + \lambda_i + \mu_q,$$

or $\pi_{ij} = \pi_{pj}$. Thus $\pi_{ij} = \pi_{i\ell}$ for every sink ℓ of G[T] and $\pi_{ij} = \pi_{pj}$ for every source p of G[T]; and since G[T] has a node $v \neq i, j$ that is both a source and a sink, it follows that $\pi_{iv} = \pi_{vj}$. Hence all π_{ij}, $(i,j) \in T \cup C_1$, are equal to some π_0, and thus (18) holds.

Finally, since every $x \in F$ has exactly $(t-1)/2$ positive components x_{ij} with $(i,j) \in T \cup C_1$, and exactly one positive component x_{ij} for every $i \in N$ and every $j \in N$, substituting into $\alpha x = \alpha_0$ the expression for α_{ij} given by (18) yields

$$\begin{aligned}\alpha_0 &= \sum_{(i,j)\in T\cup C_1} (\lambda_i+\mu_j+\pi_0)x_{ij} + \sum_{(i,j)\notin T\cup C_1} (\lambda_i+\mu_j)x_{ij} \\ &= \sum(\lambda_i : i\in N) + \sum(\mu_j : j\in N) + ((t-1)/2)\pi_0\end{aligned}$$

which is (19).∎

Theorem 4.2. For $n \geq 10$ and any odd CAT T incident with at least 8 and at most $n - 2$ nodes, the inequality (16) defines a facet of the TS polytope P^*.

Proof. Our proof will parallel that of Theorem 4.1, the main difference being that while interchanging a pair of appropriate indices in an assignment produces another assignment, only a triple interchange can get one from a given tour to another tour.

Let

$$F^* := P^* \cap \{x \mid x(T\cup C_1) = (t-1)/2\}.$$

We will show that $\dim F^* = \dim P^* - 1$ by proving that if $\alpha x \leq \alpha_0$ for all $x \in P^*$ and $\alpha x = \alpha_0$ for all $x \in F^*$, then $\alpha x = \alpha_0$ is a linear combination of (2) and $x(T \cup C_1) = (t-1)/2$.

W.l.o.g., assume that nodes $n - 1$ and n are not incident with T, and define λ_i, μ_j by (17). As in the proof of Theorem 4.1, we will show the existence of a scalar π_0 which together with these λ_i, μ_j, satisfies the relations (18) and (19).

For $(i,j) \notin T \cup C_1$, we have to show (18'), which is obviously true for $i = n - 1$, $j \in N - \{n - 1\}$, and for $i \in N - \{n\}$, $j = n$. For $i \neq n - 1$, $j \neq n$, $i \neq j$, let k, ℓ and p be distinct nodes, other than i, j, $n - 1$ and n, such that none of the arcs (k,ℓ), (p,ℓ), (p,j) belong to $T \cup C_1$. Consider $x, \bar{x} \in F^*$ such that $x_{k\ell} = x_{pj} = x_{n-1,n} = 1$, $\bar{x}_{k\ell} = \bar{x}_{pj} = \bar{x}_{in} = 1$, and the tour defined by x (by $\bar{x}$) traverses the arc (k,ℓ) after (p,j) and before $(n-1,n)$ (before i,n)). Since G[T] has τ nodes, with $8 \leq \tau \leq n - 2$, such tours always exist (see Figure 7 for an illustration). Now define x' and $\bar{x}'$ by $x'_{k\ell} = x'_{pj} = x'_{n-1,n} = 0$, $x'_{kn} = x'_{p\ell} = x'_{n-1,j} = 1$, $x'_{rs} = x_{rs}$ for all other r, s; and $\bar{x}'_{k\ell} = \bar{x}'_{pj} = \bar{x}'_{in} = 0$, $\bar{x}'_{kn} = \bar{x}'_{p\ell} = \bar{x}'_{ij} = 1$, $\bar{x}'_{rs} = \bar{x}_{rs}$ for all other r, s. Then x', $\bar{x}'$ define tours, each of which has the same number of arcs in $T \cup C_1$ as x and $\bar{x}$; hence $x', \bar{x}' \in F^*$. By assumption, we then have $\alpha x = \alpha_0 = \alpha x'$ and $\alpha\bar{x} = \alpha_0 = \alpha\bar{x}'$. Thus

$$\alpha x - \alpha x' = \alpha_{k\ell} + \alpha_{pj} + \alpha_{n-1,n} - \alpha_{kn} - \alpha_{p\ell} - \alpha_{n-1,j} = 0 ,$$

$$\alpha\bar{x} - \alpha\bar{x}' = \alpha_{k\ell} + \alpha_{pj} + \alpha_{in} - \alpha_{kn} - \alpha_{p\ell} - \alpha_{ij} = 0 ,$$

and subtracting the two equations yields

$$\alpha_{ij} = \alpha_{in} + \alpha_{n-1,j} - \alpha_{n-1,n}$$

as required by (18').

For $(i,j) \in T \cup C_1$, we introduce multipliers π_{ij} defined as in (20), and then show that they are all equal to some π_0. For this purpose, let

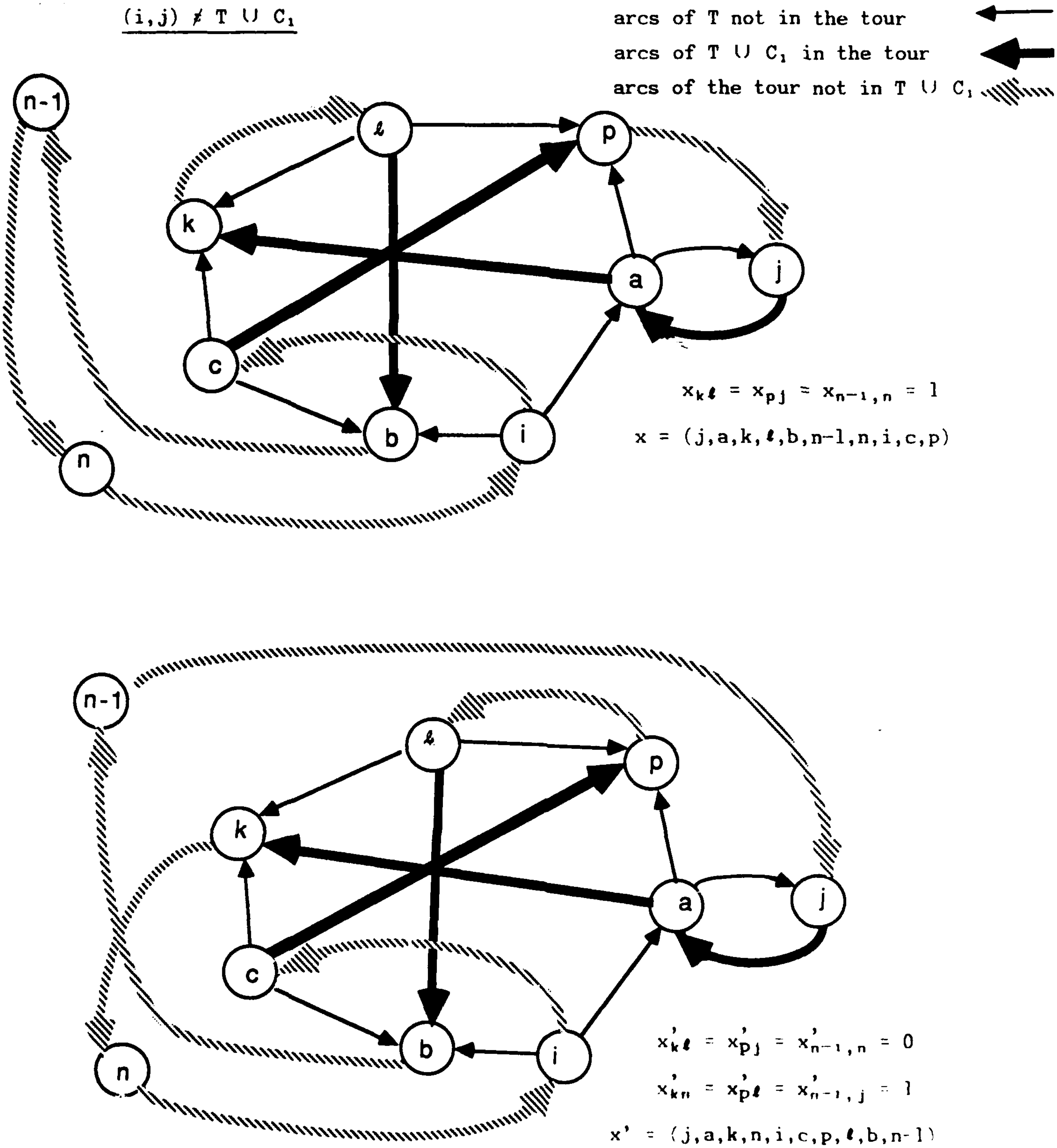

(i,j) ∉ T ∪ C1
arcs of T not in the tour
arcs of T ∪ C1 in the tour
arcs of the tour not in T ∪ C1
n-1
ℓ
p
k
a
j
c
b
i
n
x_kℓ = x_pj = x_n-1,n = 1
x = (j,a,k,ℓ,b,n-1,n,i,c,p)
n-1
ℓ
p
k
a
j
c
b
i
n
x'_kℓ = x'_pj = x'_n-1,n = 0
x'_kn = x'_pℓ = x'_n-1,j = 1
x' = (j,a,k,n,i,c,p,ℓ,b,n-1)

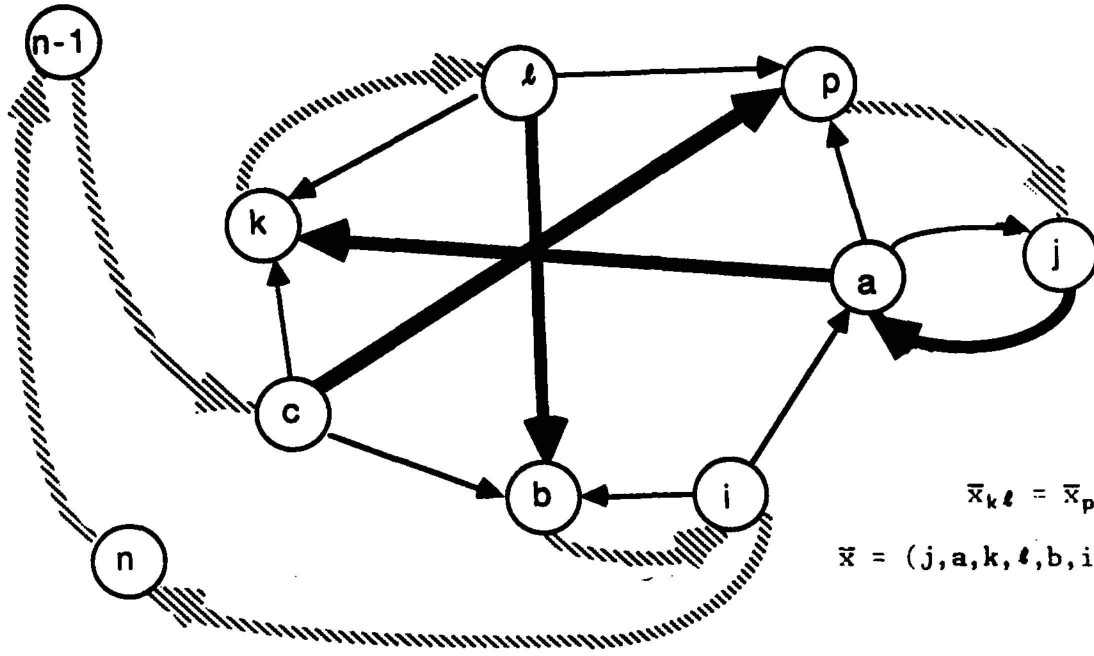
Figure 7a

$\underline{(i,j) \notin T \cup C_1}$

arcs of T not in the tour

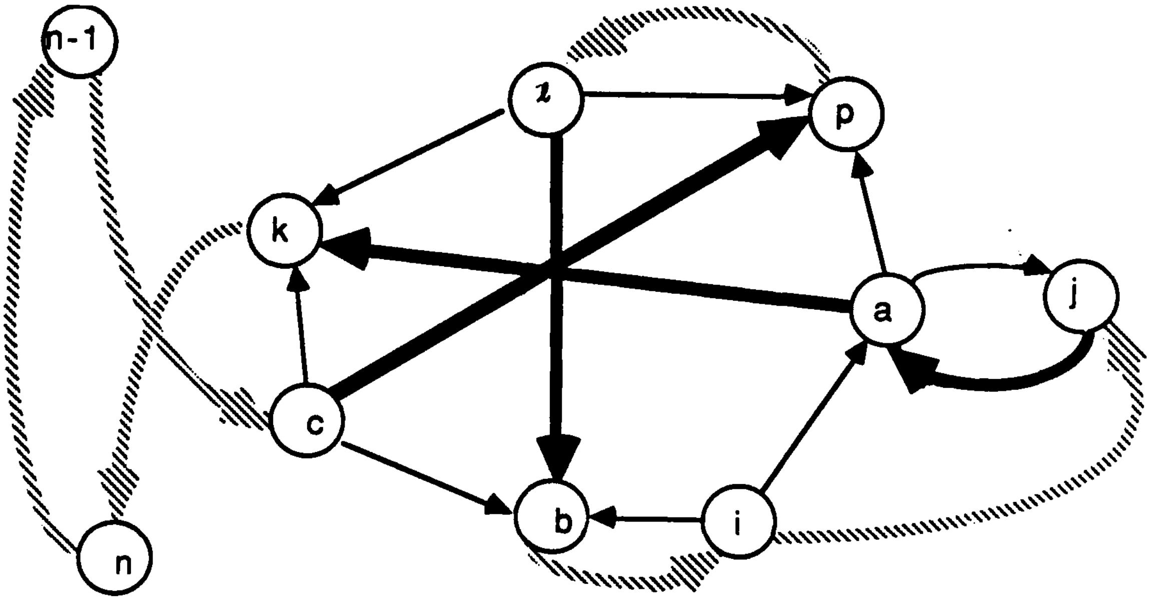
arcs of $T \cup C_1$ in the tour

arcs of the tour not in $T \cup C_1$



$\bar{x}_{k\ell} = \bar{x}_{pj} = \bar{x}_{in} = 1$

$\bar{x} = (j,a,k,\ell,b,i,n,n-1,c,p)$



$\bar{x}'_{k\ell} = \bar{x}'_{pj} = \bar{x}'_{in} = 0$

$\bar{x}'_{kn} = \bar{x}'_{p\ell} = \bar{x}'_{ij} = 1$

$\bar{x}' = (j,a,k,n,n-1,c,p,\ell,b,i)$

Figure 7b

$(i,j) \in T \cup C_1$ be such that i is a source and j is a sink of $G[T]$, and there exists a node v of $G[T]$, $i \neq v \neq j$, that is both a source and a sink.

For every sink q of $G[T]$, $q \neq i, j$, choose a pair of arcs k, ℓ, distinct from i, j, q, $n-1$ and n, such that ℓ is a source and $(k,j) \notin T \cup C_1$. Consider $x, \bar{x} \in F^*$ such that $x_{n-1,n} = x_{kj} = x_{\ell q} = 1$, $\bar{x}_{n-1,n} = \bar{x}_{kj} = \bar{x}_{iq} = 1$, and the tour defined by x (by $\bar{x}$) traverses $(n-1,n)$ after (k,j) and before (ℓ,q) (before (i,q)). Such $x, \bar{x}$ always exist (see Figure 8 for an illustration). Now define x' and $\bar{x}'$ by $x'_{n-1,n} = x'_{kj} = x'_{\ell q} = 0$, $x'_{n-1,q} = x'_{kn} = x'_{\ell j} = 1$, $x'_{rs} = x_{rs}$ for all other r, s; and $\bar{x}'_{n-1,n} = \bar{x}'_{kj} = \bar{x}'_{iq} = 0$, $\bar{x}'_{n-1,q} = \bar{x}'_{kn} = \bar{x}'_{ij} = 1$, $\bar{x}'_{rs} = \bar{x}_{rs}$ for all other r, s. Then x' and $\bar{x}'$ are tours. Furthermore, since $T \cup C_1$ contains exactly one of the arcs $(n-1,n)$, (k,j), (ℓ,q) (namely (ℓ,q), since ℓ is a source and q is a sink of $G[T]$) and exactly one of the arcs $(n-1,q)$, (k,n) and (ℓ,j) (namely (ℓ,j)), the tours defined by x and x' contain the same number of arcs of $T \cup C_1$. Similarly, since $T \cup C_1$ contains exactly one of the three arcs $(n-1,n)$, (k,j), (i,q) (namely (i,q)), and exactly one of the arcs $(n-1,q)$ (k,n), (i,j) (namely (i,j)), the tours defined by $\bar{x}$ and $\bar{x}'$ contain the same number of arcs of $T \cup C_1$. Hence $x', \bar{x}' \in F^*$ and $\alpha x = \alpha x' = \alpha_0 = \alpha\bar{x} = \alpha\bar{x}'$. Thus

$$\alpha_{n-1,n} + \alpha_{kj} + \alpha_{\ell q} - \alpha_{n-1,q} - \alpha_{kn} - \alpha_{\ell j} = 0 ,$$

$$\alpha_{n-1,n} + \alpha_{kj} + \alpha_{iq} - \alpha_{n-1,q} - \alpha_{kn} - \alpha_{ij} = 0 ,$$

and subtracting the two equations yields

$$\alpha_{ij} + \alpha_{\ell q} = \alpha_{iq} + \alpha_{\ell j}. \tag{23}$$

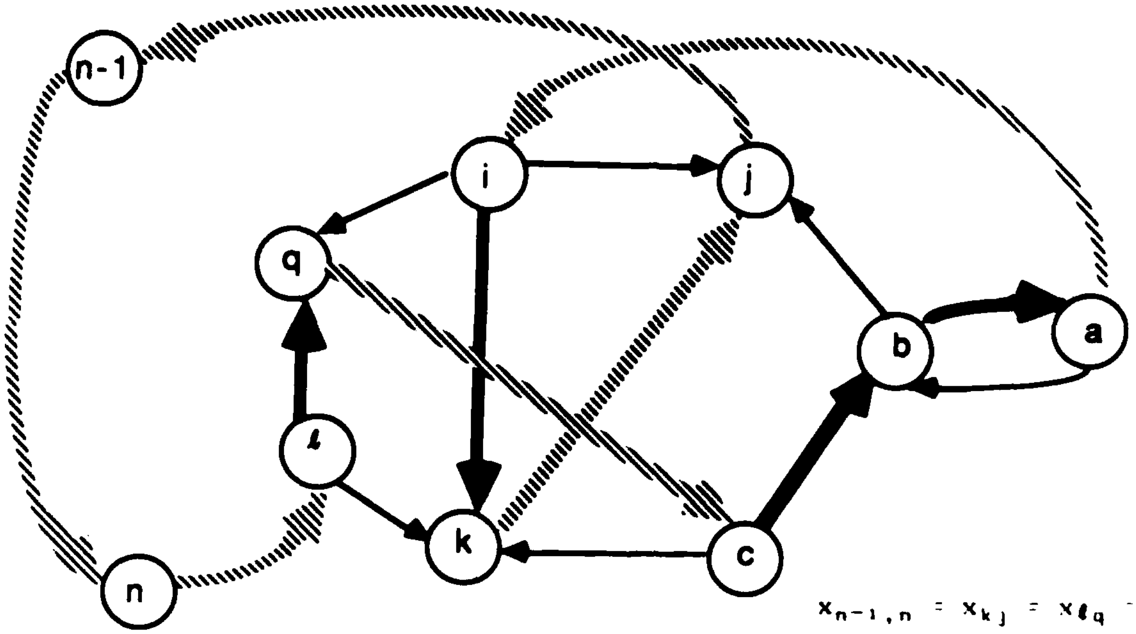
Similarly, for every source p of $G[T]$, $p \neq i, j$, one can choose a pair of nodes h, m, distinct from i, j, p, $n-1$ and n, such that m is a sink and $(i,h) \notin T \cup C_1$. Consider $x, \bar{x} \in F^*$ such that $x_{n-1,n} = x_{ih} = x_{pm} = 1$, $\bar{x}_{n-1,n} = \bar{x}_{ih} = \bar{x}_{pj} = 1$, and the tour defined by x (by $\bar{x}$) traverses (i,h) after $(n-1,n)$ and before (p,m) (before (p,j)). Now define x' by $x'_{n-1,n} = x'_{ih} = x'_{pm} = 0$, $x'_{n-1,h} = x'_{im} = x'_{pn} = 1$, $x'_{rs} = x_{rs}$ for all other r, s; and $\bar{x}'$ by $\bar{x}'_{n-1,n} = \bar{x}'_{ih} = \bar{x}'_{pj} =$

$(i,j) \in T \cup C_1$

arcs of T not in the tour

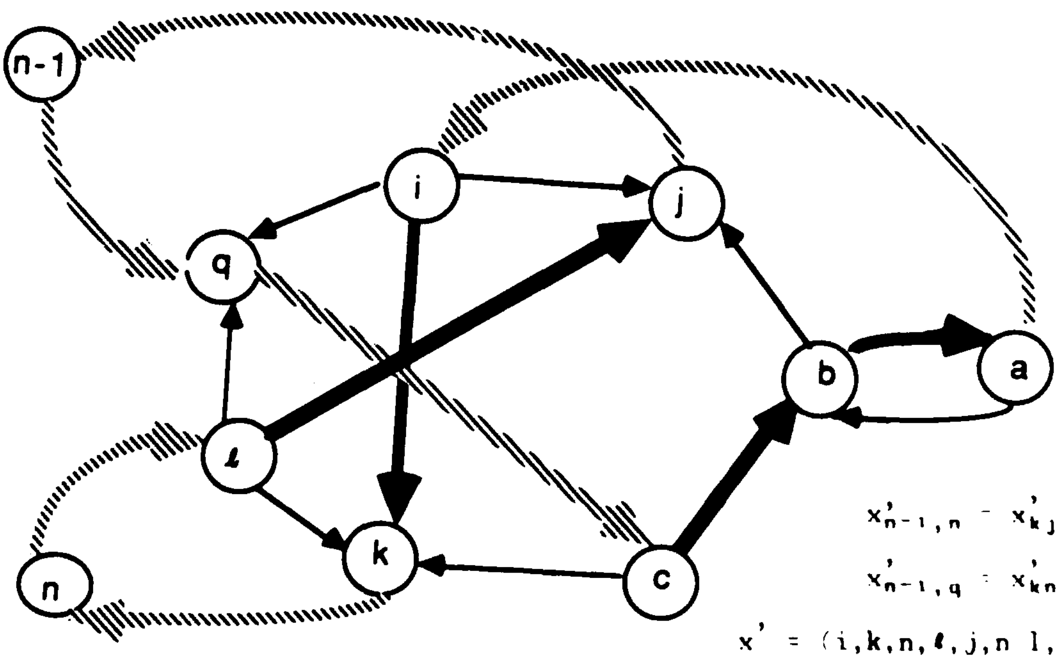
arcs of $T \cup C_1$ in the tour

arcs of the tour not in $T \cup C_1$



$x_{n-1,n} = x_{kj} = x_{\ell q} = 1$

$x = (i,k,j,n-1,n,\ell,q,c,b,a)$



$x'_{n-1,n} = x'_{kj} = x'_{\ell q} = 0$

$x'_{n-1,q} = x'_{kn} = x'_{\ell j} = 1$

$x' = (i,k,n,\ell,j,n-1,q,c,b,a)$

Figure 8a

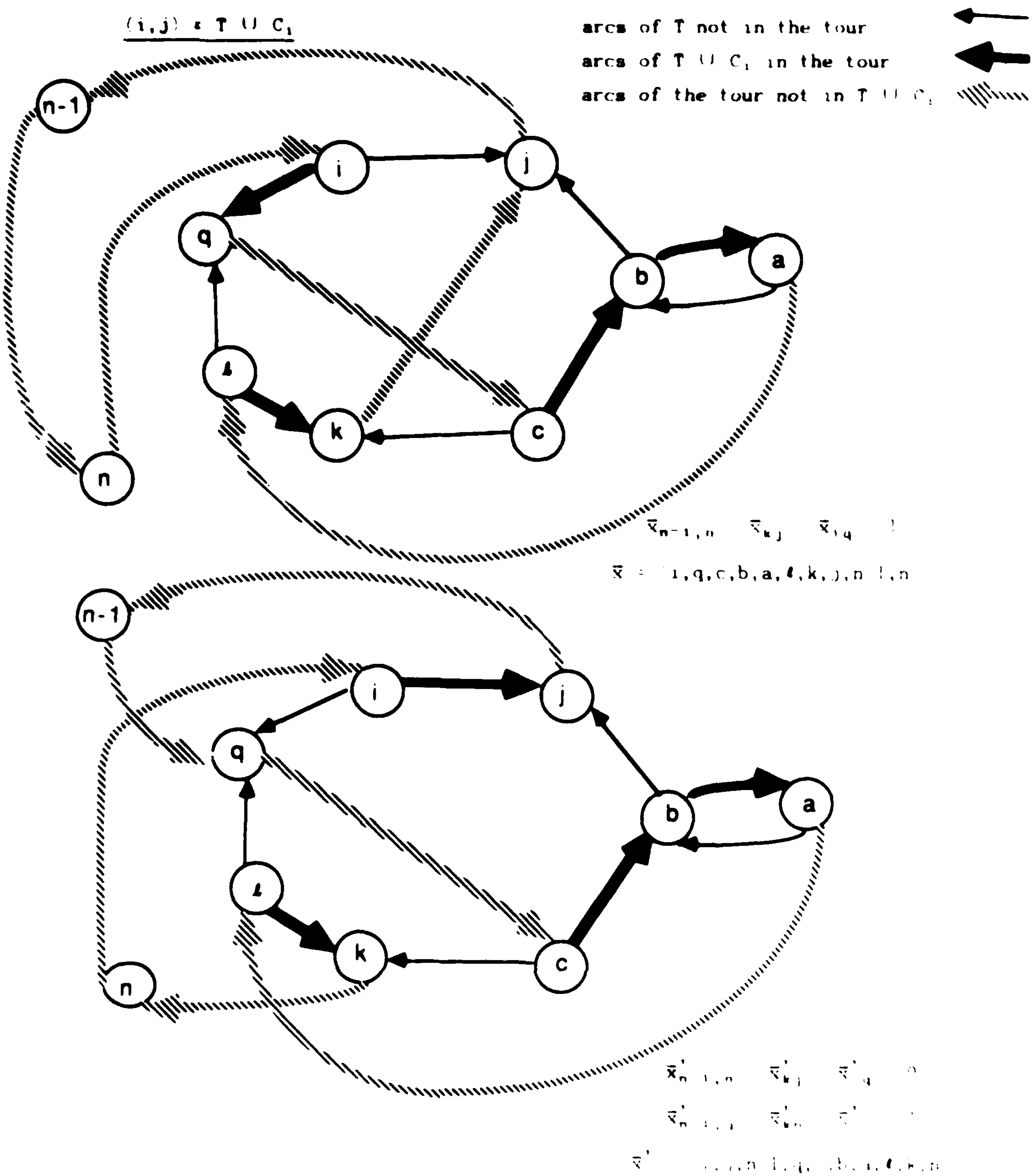

(i,j) ∈ T ∪ C$_1$
arcs of T not in the tour
arcs of T ∪ C$_1$ in the tour
arcs of the tour not in T ∪ C$_1$
n-1
i
j
q
b
a
ℓ
k
c
n
$\bar{x}_{n-1,n}$ $\bar{x}_{kj}$ $\bar{x}_{iq}$
$\bar{x}$ = (i,q,c,b,a,ℓ,k,j,n-1,n)
n-1
i
j
q
b
a
ℓ
k
c
n
$\bar{x}'_{n-1,n}$ $\bar{x}'_{kj}$ $\bar{x}'_{iq}$
$\bar{x}'_{n-1,}$ $\bar{x}'_{k}$ $\bar{x}'$
$\bar{x}'$

Figure 8b

0, $\bar{x}'_{n-1,h} = \bar{x}'_{ij} = \bar{x}'_{pn} = 1$, $\bar{x}'_{rs} = \bar{x}_{rs}$ for all other r, s. Then x' and $\bar{x}'$ are tours, and since $(p,j) \in T \cup C_1$, $(i,m) \in T \cup C_1$ (as p and i are sources and j and m are sinks of G[T]), the tours defined by x and x', contain the same number of arcs in $T \cup C_1$. Thus $x', \bar{x}' \in F^*$ and

$$\alpha_{n-1,n} + \alpha_{ih} + \alpha_{pm} - \alpha_{n-1,h} - \alpha_{im} - \alpha_{pn} = 0 ,$$

$$\alpha_{n-1,n} + \alpha_{ih} + \alpha_{pj} - \alpha_{n-1,h} - \alpha_{ij} - \alpha_{pn} = 0 .$$

Subtracting the last two equations then yields

(24) $\qquad \alpha_{ij} + \alpha_{pm} = \alpha_{pj} + \alpha_{im}$.

Then substituting into (23) and (24) the expression from (20) for α_{rs} such that $(r,s) \in T \cup C_1$ and the expression from (18) for α_{rs} such that $(r,s) \notin T \cup C_1$ yields $\pi_{ij} = \pi_{iq}$ for every sink q of G[T], and $\pi_{ij} = \pi_{pj}$ for every source p of G[T]. Since G[T] has a node $v \neq i, j$ that is both a source and a sink of G[T], it follows that $\pi_{k\ell} = \pi_0$ for all $(k,\ell) \in T \cup C_1$ and thus (18) holds.

Finally, substituting into $\alpha x = \alpha_0$ the expression for α_{ij} given by (18) yields (19).∎

The pairs of tours x, x' and $\bar{x}$, $\bar{x}'$ used in the proof of Theorem 4.2 are illustrated in Figure 7 for $(i,j) \notin T \cup C_1$ and in Figure 8 for $(i,j) \in T \cup C_1$.

5. Relation to Earlier Work

Several classes of valid, and sometimes facet defining, inequalities for the traveling salesman polytope on a directed graph are known. For a thorough survey of the relevant literature see Grötschel and Padberg [1985].

First, if

$$\sum(\alpha_{ij} y_{ij} : (i,j) \in E) \leq \alpha_0$$

is a valid inequality for the TS polytope on an undirected graph, then

$$\sum(\alpha_{ij}(x_{ij} + x_{ji}) : i < j) \leq \alpha_0$$

is a valid inequality for the TS polytope on the corresponding directed graph. Thus the various classes of facet defining inequalities for the TS polytope on an undirected graph, like the subtour eliminations inequalities (Dantzig, Fulkerson and Johnson [1954]), the comb inequalities (Chvatal [1973b], Grötschel and Padberg [1979]), the clique tree inequalities (Grötschel and Pulleyblank [1985]) have their correspondents as valid inequalities for the TS polytope on a directed graph. Whether or not these inequalities are facet defining has not yet been elucidated for every class. The subtour elimination inequalities are facet defining for P^* for $n \geq 5$ and all subtours of length ℓ, $2 \leq \ell \leq n - 2$;

the comb inequalities are facet defining for $\tilde{P}^*$ for $n \geq 6$, but whether they are facet defining for P^* is an open question (except if $n = 6$ or 7, in which case they are not). As to the more general clique tree inequalities, it is not known at this point whether they are facet defining for either $\tilde{P}^*$ or P^*.

All the above classes share the feature that the inequalities belonging to them are *symmetric* in the sense that an arc (i,j) belongs to the support of such an inequality if and only if (j,i) does. The inequalities associated with odd CAT's do not have this property, except for some special cases; so they are distinct from each of the above classes. As to those special cases, they arise when a subset of T together with the chords of type 1 form a complete digraph; in which case this complete digraph becomes the handle of a comb, whose teeth are the directed 2-cycles of T. Such is the case, for instance, with the odd CAT of length 7 shown in Figure 6 (c). More generally, all comb inequalities corresponding to combs whose handle H and teeth T_i, $i = 1, \ldots, k$ satisfy

(i) $|T_i| = 2, \qquad i = 1, \ldots, k$

and

(ii) $H \subset \left(\bigcup_{i=1}^{k} T_i \right)$

are special cases of odd CAT inequalities and thus, from Theorem 4.2, for $|H| \geq 5$ and $n \geq 12$ they define facets of P^*.

Several classes of asymmetric valid inequalities for the TS polytope on a digraph have been identified by Grötschel [1977] and Grötschel and Wakabayashi [1981 a, b]. Some of these are derived by lifting the (weak) subtour elimination inequalities. As the support of each such inequality contains a subtour, the odd CAT inequalities, whose support contains no subtour except for some special cases, are obviously distinct from this class. Other classes are associated with hypohamiltonian and hyposemihamiltonian graphs; again, these do not subsume the odd CAT inequalities. Finally, the class of so-called T_k-inequalities (Grötschel [1977]) overlap with the odd CAT inequalities for $k = 2$, in that the T_2-inequality is precisely the odd CAT inequality on four nodes depicted in Figure 5.

6. Conclusion

We have given a partial linear description of the asymmetric assignment polytope P defined on a digraph G, by identifying a family of valid inequalities associated with odd closed alternating trails of G. These inequalities are facet defining for P, and for sufficiently large graphs and sufficiently long trails, they are also facet defining for the traveling salesman polytope P^* on G.

It is to be expected that these inequalities will provide improved bounds and enhanced solution procedures for the asymmetric TSP when used as cutting planes either in the context of a pivoting algorithm like that of

Grötschel and Padberg [1985], or in the context of a Lagrangean-based algorithm that takes the cuts in the objective function with appropriate multipliers, like that of Balas and Christofides [1981].

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As assignment (spanning union of node disjoint dicycles) in a directed graph is called asymmetric if it contains at most one arc of each pair (i,j), (j,i). We describe a class of facets for the asymmetric assignment polytope, associated with certain odd length closed alternating trails. The inequalities defining these facets are also facet defining for the traveling salesman polytope on the same digraph. Furthermore, this class of facets is distinct from each of the classes identified earlier.

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